

ON THE LEVI GRAPH OF POINT-LINE CONFIGURATIONS

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ABSTRACT. We prove that the well-covered dimension of the Levi graph of a point-line configuration (v_r, b_k) is equal to 0, whenever $r > 2$.

1. INTRODUCTION

The concept of the well-covered space of a graph was first introduced in 1998 and 1999 by Caro, Ellingham, Ramey, and Yuster (see [5] and [6]) as an effort to generalize the study of well-covered graphs. Brown and Nowakowski [4], in 2005, continued the study of this object and, among other things, provided several examples of graphs featuring odd behaviors regarding their well-covered space. One of these special situations occurs when the well-covered space of the graph is trivial, i.e. when the graph is *anti-well-covered*. In this work, we prove that almost all Levi graphs of configurations (v_r, b_k) are anti-well-covered.

We start our exposition by providing the following definitions and previously-known results. Any introductory concepts failed to be defined here may be found in the books by Bondy & Murty [3] and Grünbaum [7].

We consider only simple and undirected graphs. A graph will be denoted as $G = (V(G), E(G))$, as is customary, where $V(G)$ is the set of vertices of the graph and $E(G)$ is the set of edges of the graph. Two vertices of a graph are said to be *adjacent* if they are connected by an edge. An *independent* set of vertices is one in which no two vertices in the set are adjacent. If an independent set, M , of a graph G , is not a proper subset of any other independent set of G , then M is a *maximal independent set* of G .

Definition 1. Let G be a graph and \mathbf{F} a field.

- (1) A function $f : V(G) \rightarrow \mathbf{F}$ is said to be a *weighting* of G . If the sum of all weights is constant for all maximal independent sets of G , then the weighting is a *well-covered weighting* of G .
- (2) The \mathbf{F} -vector space consisting of all well-covered weightings of G is called the *well-covered space* of G (relative to \mathbf{F}).
- (3) The dimension of this vector space is called the *well-covered dimension* of G , denoted $wcdim(G, \mathbf{F})$.

Remark 1. For some graphs, the characteristic of the field \mathbf{F} makes a difference when calculating the well-covered dimension (see [2] and [4]). If $\text{char}(\mathbf{F})$ does not cause a change in the well-covered dimension, then the well-covered dimension is denoted as $wcdim(G)$.

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In order to calculate the well-covered dimension of a graph, G , one would generally need to find all possible maximal independent sets of G . However, finding all maximal independent sets is not always an easy task, as this is a known NP-complete problem.

Despite the NP-complete nature of this problem, let us assume that we have found all possible maximal independent sets of G . We will denote these maximal independent sets as M_i for $i = 0, 1, \dots, k-1$. The well-covered weightings of G are determined by solving a system of linear equations that arise from considering all equations of the form $M_0 = M_i$ for $i = 1, \dots, k-1$. When subtraction occurs in each of these equations, we get a homogeneous system. We can then form an associated matrix, A_G from this homogeneous system of linear equations. It follows that the nullity of A_G will be the dimension of the well-covered space of G . Thus,

$$wcdim(G, \mathbf{F}) = |V(G)| - \text{rank}(A_G).$$

We now move onto another component of our work: configurations.

Definition 2 (Grünbaum [7]). *A configuration is a family of points and lines that satisfy these conditions.*

- (1) *It is a symmetric relation.*
- (2) *Incidence is only between a single point and line.*
- (3) *Two points are incident with at most one line.*
- (4) *Two lines are incident with at most one point.*

Next, there is some notation for configurations that needs to be set, as well as specific parameters that need to be established for the main result of this work.

Definition 3. *We define a (v_r, b_k) configuration as a point-line configuration such that*

- (1) *There are exactly k points incident with each line, and $k \geq 2$.*
- (2) *There are exactly r lines incident with each point, and $r \geq 2$.*
- (3) *There are exactly v points in (v_r, b_k) , and $v \geq 4$.*
- (4) *There are exactly b lines in (v_r, b_k) , and $b \geq 4$.*

When $v = b$ and $r = k$, the configuration will be denoted by (v_r) .

Example 1. *Several well-known geometric structures fall into the category of (v_r, b_k) configurations. For instance:*

- (1) *A projective plane of order q is a $(q^2 + q + 1_{(q+1)})$ configuration, where q is the power of a prime. See Figure 1 for a representation of $PG(2, 3) = (13_4)$.*
- (2) *The Pappus configuration is a (9_3) configuration, and the Desargues configuration is a (10_3) configuration.*
- (3) *$PG(n, q)$ is a $\left(\frac{q^{n+1} - 1}{q - 1}_{(q+1)}, \frac{(q^{n+1} - 1)(q^n - 1)}{(q^2 - 1)(q - 1)}_{(q^2 + q + 1)} \right)$ configuration, where q is the power of a prime.*
- (4) *A generalized quadrangle $G(s, t)$ is a $((1+s)(st+1)_{(1+s)}, (1+t)(st+1)_{(1+t)})$ configuration.*

The reader is referred to the book by Batten [1] for more information about these important geometric objects.

Finally, we define the object, Levi graphs, that will connect configurations and graphs.

Definition 4. Given a configuration, \mathcal{C} , we define the Levi graph of \mathcal{C} , denoted $Levi_{\mathcal{C}}$, as the bipartite graph with vertices given by the points and lines in \mathcal{C} . The edges of $Levi_{\mathcal{C}}$ connect a point-vertex P with a line-vertex ℓ if and only if the point P is incident with the line ℓ in \mathcal{C} . No other edges exist in $Levi_{\mathcal{C}}$.

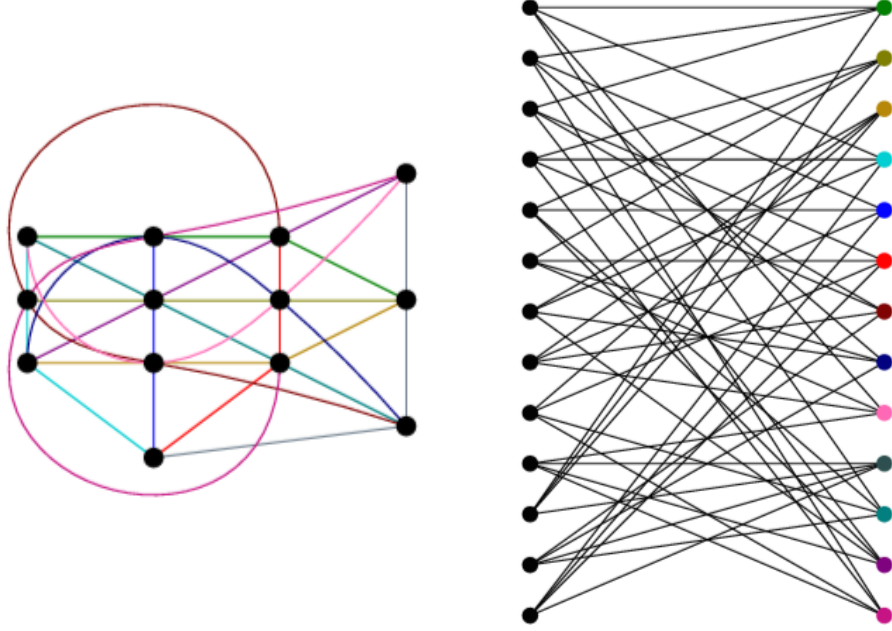


FIGURE 1. $(13_4) = PG(2, 3)$ and $Levi_{(13_4)}$

Our main result, which will be proven in the following section, combines all these objects as follows:

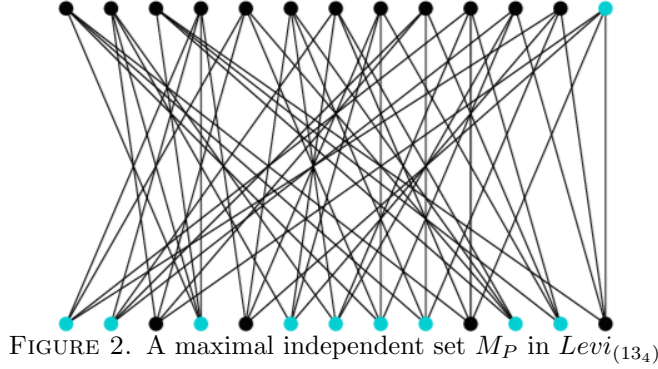
Theorem 1. *If $r \in \mathbb{N}$ and $r > 2$, then $wcdim(Levi_{(v_r, b_k)}) = 0$.*

2. THE WELL-COVERED DIMENSION OF $Levi_{(v_r, b_k)}$

We will prove Theorem 1 by first proving a technical lemma, that introduces a family of maximal independent sets that will show to be useful later on.

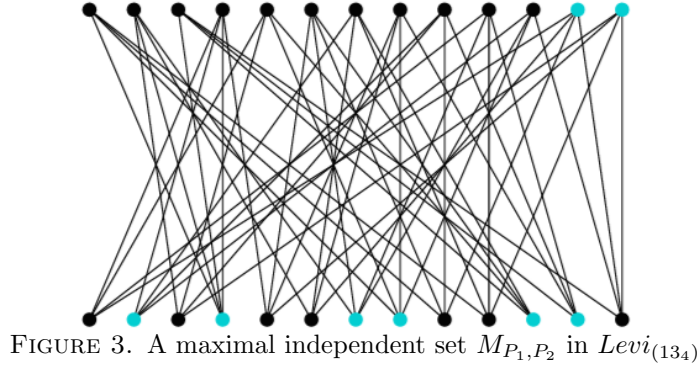
Lemma 1. *A Levi graph of a configuration (v_r, b_k) , where $r > 2$, has at least $v + b + 2$ maximal independent sets.*

Proof. Let P be a fixed point in (v_r, b_k) . We consider the set, M_P , of vertices of $Levi_{(v_r, b_k)}$ given by P and all the lines not incident to P . This is an independent set of $Levi_{(v_r, b_k)}$ because there is no incidence between vertices in the set. Moreover, note that if we included another point-vertex to M_P , then that vertex would be adjacent to one of the line-vertices in M_P (because of condition (2) in Definition 2, and the fact that $r > 2$). Also, if another line-vertex were to be added to M_P , then this line would have to be incident with P . It follows that M_P is a maximal independent set of $Levi_{(v_r, b_k)}$.

FIGURE 2. A maximal independent set M_P in $Levi_{(13_4)}$

Repeating this construction for all v points in (v_r, b_k) , we get v distinct maximal independent sets of $Levi_{(v_r, b_k)}$.

We will now construct another b distinct maximal independent sets of $Levi_{(v_r, b_k)}$. We start by fixing a line ℓ in (v_r, b_k) and then any two distinct points $P_1, P_2 \in \ell$ (recall that $k \geq 2$). We consider the set, M_{P_1, P_2} of vertices of $Levi_{(v_r, b_k)}$ given by P_1, P_2 and all the lines not incident to either of these points. Note that this forms an independent set since adjacency in $Levi_{(v_r, b_k)}$ only occurs if incidence occurs in (v_r, b_k) . If we try to add in another vertex-point to M_{P_1, P_2} , since $r > 2$, then this point will be incident to one of the lines not through P_1 or P_2 and will therefore be adjacent to the vertex-lines in M_{P_1, P_2} . If we try to add another vertex-line to M_{P_1, P_2} , then this line will be incident to one or both P_1 and P_2 . Therefore, M_{P_1, P_2} is a maximal independent of $Levi_{(v_r, b_k)}$.

FIGURE 3. A maximal independent set M_{P_1, P_2} in $Levi_{(13_4)}$

Repeating this construction for all b lines in (v_r, b_k) (it does not matter what pair of points one picks on any given line), we get b distinct maximal independent sets of $Levi_{(v_r, b_k)}$.

Finally, note that the set of all point-vertices in $Levi_{(v_r, b_k)}$ is a maximal independent set as well as the set of all line-vertices in $Levi_{(v_r, b_k)}$. Hence, we have constructed $v + b + 2$ distinct maximal independent sets in $Levi_{(v_r, b_k)}$. \square

Next, we proceed to prove our main result.

Proof of Theorem 1. We denote by \mathbf{F} the field of scalars of the well-covered space of $G = \text{Levi}_{(v_r, b_k)}$, where $r > 2$. Let A_G be the associated matrix of G , and note that A_G has $v+b$ columns. In order to prove that A_G has $v+b$ linearly independent rows we will consider the $v+b+2$ maximal independent sets in Lemma 1.

We create the first v rows of A_G by equating the weight of each of the maximal independent sets M_P to the weight of the maximal independent set consisting of all the lines of G . After subtracting we obtain v equations of the form

$$(1) \quad f(P) - f(\ell_1) - f(\ell_2) - \cdots - f(\ell_r) = 0$$

where each ℓ_i is incident with P . It follows that, after organizing the columns of A_G by putting point-vertices first and then line-vertices, the ‘first’ v rows of A_G are

$$\begin{bmatrix} I_v & -C \end{bmatrix}$$

where C is the incidence matrix of $\text{Levi}_{(v_r, b_k)}$.

In order to obtain the next b rows of A_G we will consider maximal independent sets of the form $M_{P,Q}$. For any given line ℓ of (v_r, b_k) , we choose (any) two points on it. We will denote these two points as P_1 and P_2 . We then consider the maximal independent set M_{P_1, P_2} and equate its weight to the weight of the maximal independent set M_{P_1} . After subtracting we obtain an equation of the form

$$(2) \quad f(P_2) - f(\ell_1) - f(\ell_2) - \cdots - f(\ell_r) + f(\ell) = 0$$

where each ℓ_i is incident with P_2 .

Note that subtracting Equation 1 (with $P = P_2$) from Equation 2 yields $f(\ell) = 0$. Since ℓ was arbitrary, we get $f(\ell) = 0$ for every line in (v_r, b_k) . It follows that since subtracting equations is just a different way to describe row operations in A_G , we get that the ‘first’ $v+b$ rows of A_G (after a few row operations) are

$$\begin{bmatrix} I_v & -C \\ \mathbf{0} & I_b \end{bmatrix}$$

Note that addition and subtraction where the only two (row) operations needed to obtain the matrix above. Hence, the first $v+b$ rows of A_G do not change depending on the characteristic of \mathbf{F} .

Since the determinant of the matrix above is non-zero, the rank of A_G is maximal, and thus $\text{wcdim}(\text{Levi}_{(v_r, b_k)}) = 0$. \square

3. POSSIBLE GENERALIZATIONS

In this section, we study possible generalizations of Theorem 1. This will be done by providing a few results and by introducing objects for which this theorem could be extended to. We begin by proving that Theorem 1 cannot be extended to configurations having exactly two lines being incident with every point. This will be done by an example that considers (v_2) configurations.

We first notice that a (v_2) configuration is a disjoint union of polygons/cycles. This is convenient because disjoint unions of graphs behave well with respect to the well-covered dimension. In fact, Lemma 5 in [4] says

$$\text{wcdim}(G \cup H) = \text{wcdim}(G) + \text{wcdim}(H),$$

where \cup stands for disjoint union.

Since we know that $\text{Levi}_{C_n} = C_{2n}$, we get the following lemma.

Lemma 2. *Let \mathcal{C} be a (v_2) configuration. Then,*

$$\mathcal{C} = \bigcup_{i=1}^t C_{n_i},$$

where $n_i > 2$, for all $1 \leq i \leq t$. Moreover,

$$wcdim(Levi_{\mathcal{C}}) = \sum_{i=1}^t wcdim(C_{2n_i})$$

Finally, we notice that Theorem 5 in [2] implies

$$wcdim(C_{2n}) = \begin{cases} 2 & \text{if } n = 3 \\ 0 & \text{if } n \geq 4 \end{cases}$$

Next is an immediate corollary of Theorem 5 in [2] and Lemma 2.

Corollary 1. *$wcdim(Levi_{\mathcal{C}})$ is even, for all (v_2) configurations \mathcal{C} . Moreover, for every $n \in \mathbb{N}$, there is a (v_2) configuration, \mathcal{C}_n , such that*

$$wcdim(Levi_{\mathcal{C}_n}) = 2n$$

In particular, the sequence $\{wcdim(Levi_{\mathcal{C}_n})\}_{n=1}^{\infty}$ is unbounded.

We conclude that Theorem 1 cannot be expanded to the case $r = 2$. However, it is still an open problem to find the well-covered dimension of all Levi graphs of (v_2, b_k) configurations.

Of course, the study of the well-covered dimension of Levi graphs of configurations not of the form (v_r, b_k) is also an interesting open problem.

Block designs are another family of objects that could be studied to attempt a generalization of Theorem 1.

Definition 5. *Let $\lambda, t \geq 1$. A $t - (v, k, \lambda)$ design (or t -design), is an incidence structure of points and blocks with the following properties:*

- (1) *there are v points;*
- (2) *each block is incident with k points;*
- (3) *any t points are incident with λ common blocks.*

It is easy to see that a $1 - (v, k, \lambda)$ design is a (v_{λ}, b_k) configuration, where $b = v\lambda/k$. Moreover, a $2 - (v, k, 1)$ design is a configuration in which every pair of points are ‘collinear.’ For $t > 1$ and $\lambda > 1$, the obvious definition of the Levi graph of a t -design would yield a multigraph. This apparent setback is not so much of a problem since having one edge or multiple edges between two vertices would mean the same when looking for maximal independent sets. We claim that the ideas used to prove Theorem 1 can be generalized to be applicable to block designs.

Finally, in this work, we study the well-covered space of the Levi graph of any given configuration. We propose, as an interesting open problem, the study of configurations via understanding the well-covered spaces of their collinearity graphs (in which points in a configuration are defined as vertices, and adjacency occurs if and only if the points are collinear). The third author is currently working on a particular case of this problem: generalized quadrangles.

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